

Homework 8

A. Chorin

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Due March 30.

1. Show that if $\phi(\lambda)$ is the characteristic function of η , then $E[\eta^n] = (-i)^n (d^n/d\lambda^n)\phi(0)$ provided both sides of the equation make sense. Use this fact to show that if $\xi_i, i = 1, \dots, n$ are Gaussian variables with means zero, NOT NECESSARILY INDEPENDENT, then $E[\xi_1 \xi_2 \cdots \xi_n] = \sum \Pi E[\xi_{i_k} \xi_{j_k}]$ for n even, and $= 0$ for n odd. In the right hand side i_k, j_k are 2 of the indices, the product is over a partition of the n indices into disjoint groups of 2, and the sum is over all such partitions (this is "Wick's theorem").
2. Consider the random differential equation $dq(t)/dt = -ibq(t), q(0) = 1$, where b is a random variable and $i = \sqrt{-1}$; define $Q(t) = E[q(t)]$. Show that $|Q(t)| \leq Q(0)$. Suppose the distribution of b is not known but you know the moments of b , i.e., you know $E[b^j]$ for all $j \leq N$. Solve the equation by iteration: $q_0 = 1, q_{j+1} = 1 - ib \int_0^t q_j(s) ds$ and then define $Q_j = E[q_j]$ for $j \leq N$, thus using the information you have. Show that however big N may be, as long as it is finite the approximation Q_j will violate the inequality $|Q(t)| \leq Q(0)$ for t large and any $j > 1$.
3. Continue the example of data assimilation from the notes: Suppose $x(t) = x(0)$ is a scalar and you have observations $y_i = x_i + gW_i$, where g is a fixed constant. What is $\hat{x}_i = E[x_i | \bar{y}]$ for $i > 1$?
4. Show that for any given wide sense stationary stochastic process with a finite variance, there exists a gaussian process with the same covariance.
5. Consider the time-series prediction example in the notes, where the covariance function is Ca^T for $T > 0$. Suppose $m = 2$, i.e., you are trying to make a prediction two steps ahead of your last observation. What is the best you can do?
6. Consider the following functions $R(T)$; which ones are the covariance functions of some stationary stochastic process, and why? ($T = t_2 - t_1$ as usual): (i) $R(T) = e^{-T^2}$; (ii) $R = Te^{-T^2}$; (iii) $R = e^{-T^2/2}(T^2 - 1)$, (iv) $R = e^{-T^2/2}(1 - T^2)$.